# THE GROUPS OF BRANCHED TWIST-SPUN KNOTS

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Abstract. We characterize the groups of branched twist spins of classical knots in terms of 3-manifold groups, and also give a purely algebraic, conjectural characterization in terms of $PD_3$-groups. We show also that each group is the group of at most finitely many branched twist spins.

In this note we shall give a characterization of the groups of branched twist spins in terms of 3-manifold groups (Theorem 2), and then give an alternative, purely algebraic formulation, which depends on the assumption that every $PD_3$-group is the fundamental group of an aspherical closed 3-manifold. The constructive aspect of the argument derives from the work of S. P. Plotnick [17, 19]. We show also that if two fibred knots have irreducible fibre and isomorphic groups then the knot manifolds are homeomorphic, unless the fibres are lens spaces (Theorem 1), and that if the group of a 2-knot $K$ is isomorphic to the group of a branched twist spin $\tau_{m,s}k$ of a prime knot $k$ then the knot manifolds are $s$-cobordant (Corollary 5). Finally, we show that each group is the group of at most finitely many branched twist spins (Theorem 12). The latter result extends the work of M. Fukuda and M. Ishikawa, who show that if $k_1$ and $k_2$ are distinct knots and $m_1 \neq m_2$ then $\tau_{m_1,s_1}k_1$ and $\tau_{m_2,s_2}k_2$ are distinct (for any $s_1, s_2$) [6], and that if $k_1$ and $k_2$ are distinct prime knots then in most cases $\tau_{m,s}k_1$ and $\tau_{m,s}k_2$ are distinct [7].

## 1. terminology

We review briefly some knot-theoretic terminology used here. (See [9, Chapter 14] for more details.)

An *n-knot* is a locally flat embedding $K : S^n \to S^{n+2}$. (Our interest here is in the cases $n = 1$ and 2.) The *exterior* $X(K)$ is the closed complement of a tubular neighbourhood of $K(S^n)$ in $S^{n+2}$, and $\partial X(K) \cong S^n \times S^1$. The *knot group* is $\pi K = \pi_1(X(K))$. If we fix orientations for the spheres then the boundary of a disc $D^2$ transverse to $K$ determines a well-defined conjugacy class of *meridians*. The knot

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group is normally generated by the image of the meridians. The *knot manifold* $M(K) = X(K) \cup D^{n+1} \times S^1$ is the closed $(n+2)$-manifold obtained by elementary surgery on the knot. (If $n = 1$ we require that $K$ have framing 0.) If $K_1$ and $K_2$ are two $n$-knots then we shall write $K_1 \cong K_2$ if there is a self-homeomorphism $h$ of $S^{n+2}$ such that $h(K_1(S^n)) = K_2(S^n)$. (We use this equivalence relation rather than ambient isotopy because the invariant of primary interest here is the knot group, which does not reflect the orientations.)

A *weight element* in $\pi K$ is an element which normally generates the group, and a *weight class* is the conjugacy class of a weight element. In general, a knot group may have many weight classes. If $n > 1$ then $\pi_1(M(K)) \cong \pi K$ and each weight class determines an isotopy class of simple closed curves in $M(K)$. Elementary surgery on such a curve gives a homotopy $(n+2)$-sphere $X(K) \cup_f S^n \times D^2$, and the cocore $S^n \times \{0\}$ of the surgery is an $n$-knot. However there are two possible framings for this surgery, and the corresponding knots $K$ and $K^*$ may be distinct. We call the knot $K^*$ the "Gluck reconstruction" of $K$, and say that $K$ is *reflexive* if $K \cong K^*$. Thus we may recover $K$ from $M(K)$ and the weight class of a meridian, up to an ambiguity of order $\leqslant 2$.

Knot groups have abelianization $\mathbb{Z}$, and so every knot group $\pi$ is a semidirect product $\pi' \rtimes_\theta \mathbb{Z}$, where $\pi'$ is the commutator subgroup. The characteristic map $\theta$ is induced by conjugation by a meridian, and the image of $\theta$ on the outer automorphism group $Out(\nu)$ depends only on the weight class. Two knot groups $\nu \rtimes_\sigma \mathbb{Z}$ and $\nu \rtimes_\theta \mathbb{Z}$ with isomorphic commutator subgroup $\nu$ are isomorphic if and only if the images of $\sigma$ and $\theta$ in $Out(\nu)$ are conjugate, up to inversion [17].

If $K$ is a knot then the proper invariant determined by the meridians is the *weight orbit*, which is the orbit of a meridian under the action of $Aut(\pi K)$ [9, 14.§7]. (Different isomorphisms between the same pair of groups may not carry a given element to conjugate elements, but the images will agree up to the action of an automorphism of the target group.) Let $\theta$ be the automorphism of $\pi'$ induced by conjugation by a weight element $t$ in $\pi$. Then the index 2 subgroup of automorphisms which induce the identity on $\pi/\pi'$ is

$$Aut^+(\pi K) = \{(g, d) \in \pi' \rtimes Aut(\pi') \mid d\theta d^{-1}\theta^{-1} = c_g\}$$

where $c_g(x) = gxg^{-1}$ for $x \in \nu$. The automorphism $(g, d)$ acts on $\pi'$ through $d$ and sends $t$ to $gt$. Each strict weight orbit contains at most $|Out(\nu)|$ weight classes. In many cases considered in this note the knot group determines the knot manifold up to $s$-cobordism, and the knot group together with the weight orbit determines the knot exterior up to relative $s$-cobordism.

Let $M_m(k)$ be the $m$-fold cyclic branched cover of $S^3$, branched over a knot $k$. If $k$ is prime and $m \geqslant 3$ then $M_m(k)$ is generically aspherical, with the following exceptions. If $m = 3, 4$ or $5$ and $k$ is the trefoil knot $3_1$ then $\nu = \pi_1(M_m(k)) \cong Q(8)$, $T_1^*$ or $I^*$, respectively, and if $m = 3$ and $k$ is the (2,5)-torus knot $5_1$ and then $\nu \cong I^*$. If $m = 3$ and $k$ is the figure eight knot $4_1$ then $M_m(k)$ is the Hantzsche-Wendt flat 3-manifold. Otherwise, $M_m(k)$ is an $\widetilde{\mathbb{SL}}$-manifold if $k$ is a torus knot, is hyperbolic if $k$ is simple but not a torus knot (and $(k, m) \neq (4_1, 3)$), and is Haken if $k$ is a satellite knot [3].

The 2-fold branched covers of Montesinos knots are Seifert fibred, but the covering involution acts non-trivially on the fibre [15]. In particular, if $k$ is a 2-bridge knot then $M_2(k)$ is a lens space. The other elliptic 3-manifolds (excepting those with group $Q(8) \times \mathbb{Z}/d\mathbb{Z}$ for some $d > 1$) are 2-fold branched covers of certain pretzel knots [15]. In all cases, $M_m(k)$ is either aspherical or has universal cover $S^3$, and $\nu$ is either a $PD_3$-group or is finite.

All the 2-knot groups with finite commutator subgroup (excepting $\pi' \cong Q(8) \times \mathbb{Z}/d\mathbb{Z}$, $d > 1$) are realized by 2-twist spins of 2-bridge knots or pretzel knots [23]. The 2-twist spins of Montesinos knots with infinite commutator subgroup $\pi'$ have abelian normal subgroups of rank 2, which are not central.

Let $\zeta G$ denote the centre of a group $G$.

## 2. FIBRED KNOTS

An $n$-knot $K$ is *fibred* if there is an $(n + 1)$-manifold $F$ (the fibre) with boundary $\partial F = S^n$ and a self-homeomorphism $c : F \to F$ (the monodromy) such that the knot exterior $X(K)$ is the mapping torus

$$MT(c) = F \times [0, 1]/(f, 1) \sim (c(f), 0), \ \forall f \in F.$$

The *closed fibre* of the knot is $\hat{F} = F \cup D^{n+1}$, and the *closed monodromy* $\hat{c}$ is the extension of the monodromy $c$ to $\hat{F}$, obtained by coning off the boundary. Thus $\hat{c}$ has a fixed point, which determines a section $S^1 \subset MT(\hat{c})$. If $K$ is fibred, with fibre $F$, then $\pi K' \cong \nu = \pi_1(F)$ and $\pi \cong \nu \rtimes_{c_*} \mathbb{Z}$. Fibred 2-knots with monodromy of order 2 are determined by their complements, but no fibred 2-knot with monodromy of finite odd order $m > 1$ is reflexive [19, Theorem 6.2].

In the classical case $n = 1$ a knot $k$ is fibred if and only if $\pi k'$ is finitely generated, and then it is a free group. If $K$ is a fibred 2-knot then the fibre is determined by $\pi K'$ (up to lens space summands). Standard results of 3-manifold theory show that the knot group of a fibred 2-knot with irreducible fibre determines the knot manifold, except when $\pi'$ is finite cyclic. (The results invoked in Theorem 1 are scattered across a number of original papers, and there is no one convenient reference.) The case of twist spins of torus knots was treated in [9, Theorem 16.6].

**Theorem 1.** *If $K_1$ and $K_2$ are fibred 2-knots such that $\pi = \pi K_1 \cong \pi K_2$ and $\pi'$ is indecomposable but not finite cyclic then $M(K_1) \cong M(K_2)$.*

*Proof.* Let $N$ be the closed fibre of $K_1$. Then $\pi_1(N) \cong \pi'$, and so $N$ is either aspherical or is a quotent of $S^3$. If $N$ is aspherical it is either Haken, hyperbolic or Seifert fibred. In each case, the homotopy class of the monodromy of an $N$-bundle over $S^1$ is determined by its image in $Out(\pi')$, and homotopic self-homeomorphisms of $N$ are isotopic. (See [16] and [8] for hyperbolic 3-manifolds, [22] and [24] for Haken 3-manifolds, [21] and [1] for the infinite Seifert fibred cases, and [14] for the cases with $\pi'$ finite.) □

If $\pi' \cong *^r G_i$ where $G_i$ has one end then the kernel of the natural map from the group of homotopy classes of homotopy self-equivalences of $N$ to $Out(\pi')$ has order $2^{r-1}$ [13], and so the homotopy class of the monodromy is not determined by its image in $Out(\pi')$ if $r > 1$. Moreover, "homotopy implies isotopy" may no longer hold [4].

## 3. THE GROUPS OF BRANCHED TWIST SPINS

Let $\tau_{m,s}k$ be the $s$-fold branched cyclic cover of the $m$-twist spin $\tau_m k$. (We shall refer to such 2-knots $\tau_{m,s}k$ as "branched $m$-twist spins", or just "branched twist spins", for brevity, and write $\tau_m k$ instead of $\tau_{m,1}k$.) The knot manifold $M(\tau_{m,s}k)$ is the mapping torus of the $s$th power of of a meridianal generator of the monodromy of $M_m(k)$ over $S^3$. There are $\varphi(m) = |\mathbb{Z}/m\mathbb{Z}^\times|$ possible choices for $s$ *mod* $(m)$, and clearly $M(\tau_{m,s+m}k) \cong M(\tau_{m,s}k)$. However $\tau_{m,s+m}k$ is actually $(\tau_{m,s}k)^*$ [5, 18]. (The natural framings of the normal bundles of the canonical sections of the mapping tori differ by one full rotation.) The knots $\tau_{m,-s}k$ and $\tau_{m,s}k$ are equivalent via an orientation-reversing homeomorphism. Hence these branched covers represent at most $\varphi(m)$ distinct knots.

S.P. Plotnick has shown that (modulo the 3-dimensional Poincaré Conjecture) a fibred knot is a branched twist spin if and only if the monodromy (of the fibration of the knot exterior) has finite order [19]. If $m$ is odd (and $> 1$) then $\tau_m k^* \not\cong \tau_m k$ [19]. On the other hand, $\tau_2 k$ is always reflexive.

If $k = k_{p,q}$ is the $(p,q)$-torus knot then $M_m(k)$ is the Brieskorn 3-manifold

$$M(m,p,q) = \{(u,v,w) \in S^5 \subset \mathbb{C}^3 \mid u^m + v^p + w^q = 0\},$$

and the monodromy is generated by $(u, v, w) \mapsto (\zeta_m u, v, w)$, where $\zeta_m = exp(\frac{2\pi i}{m})$ is a primitive $m$th root of unity. The map from $M(m, p, q)$ to $\mathbb{CP}^1 = S^2$ given by $(u, v, w) \mapsto [v^p : w^q]$ is invariant under the monodromy, and induces a Seifert fibration of $M(\tau_{m,s}k)$ over the orbifold $S^2(m, p, q)$, with general fibre the torus, for each $s$ with $(s, m) = 1$.

**Theorem 2.** *A knot group $\pi$ is the group of a branched $m$-twist spin of a classical knot, where $m > 1$, if and only if*

1. *there is a weight element $t$ such that $t^m$ is central;*
2. *the indecomposable factors of the commutator subgroup $\pi'$ are 3-manifold groups;*
3. *conjugation by $t$ induces an automorphism of each such factor;*
4. *if $G_i$ is an infinite factor of $\pi'$ this automorphism has order $m$;*
5. *$\pi'$ has no $\mathbb{Z}$ factor and no factor $Q(8) \times \mathbb{Z}/d\mathbb{Z}$ with $d > 1$;*
6. *if $m = 3$ the only finite factors of $\pi'$ are $Q(8)$ or $I^*$; if $m = 4$ only $T_1^*$; if $m = 5$ only $I^*$; and if $m > 5$ then $\pi$ is torsion-free.*

*Proof.* Let $k$ be a 1-knot, and let $N = M_m(k)$ be the $m$-fold cyclic branched cover of $S^3$, branched over $k$. If $N$ is aspherical then the monodromy induces an automorphism of $\nu = \pi_1(N)$ of order $m$ [2]. If $\zeta\nu = 1$ then $Inn(\nu) \cong \nu$ is torsion-free, and so the image of the monodromy in $Out(\nu) = Aut(\nu)/Inn(\nu)$ also has order $m$. If $k = \Sigma k_i$ is composite then $N$ is the connected sum of the cyclic branched covers of the summands and $\tau_{m,s}k = \Sigma\tau_{m,s}k_i$ is the corresponding sum.

The necessity of the conditions follows from the above observations. Suppose that they hold. Then $\pi'$ is finitely presentable, and so has a finite factorization into indecomposables. Conditions (3) and (4) allow us to reduce to the case when $\pi'$ is indecomposable. If $\nu = \pi'$ is finite then suitable twist spins of 2-bridge knots, pretzel knots or small torus knots give all the possible examples. If $\pi'$ has one end and $\zeta\pi' \neq 1$ then this is essentially [9, Theorem 16.5]. The rest of the proof follows as indicated in the final paragraph of [9, 16.§3]. The key point is that the group $\pi/\langle t^m\rangle \cong \pi' \rtimes \mathbb{Z}/m\mathbb{Z}$ acts on the universal covering space $\widetilde{N}$, and the image of $t$ has a connected, non-empty fixed point set, by Smith theory. Let $c$ be the self-homeomorphism of $N_o = \overline{N \setminus D^3}$ corresponding to $t$. Then $MT(c) = X(K)$ for a 2-knot $K$, and $c$ has finite order. Since the Poincaré Conjecture is now known to hold, $X(K)$ is the exterior of a branched twist spin [19, Proposition 6.1]. $\square$

**Corollary 3.** *Let $K = \tau_{m,s}k$ and $\pi = \pi K$. Then the image of a meridian for $K$ in $\pi/\zeta\pi$ has order $m$. If $\pi'$ has at least one factor which has one end and trivial centre then $m = [\pi : \pi'.\zeta\pi]$.*

*Proof.* The image of a meridian of $\tau_{m,s}k_{p,q}$ in the central quotient $\pi K/\zeta\pi K \cong \pi^{orb}(S^2(m,p,q))$ has order $m$ [9, Lemma 16.7]. If $k$ is not a torus knot then both assertions follow immediately from the first paragraph of the proof of the theorem. □

Suppose that $k_{p,q}$ is a nontrivial torus knot, and that $K = \tau_{m,s}k_{p,q}$ is also an $m'$-twist spin $K = \tau_{m',s'}k'$ for some other knot $k'$ and order of twisting $m'$. Then $k'$ is also a torus knot, since $\zeta\pi K$ is not cyclic, and the images of the meridians for $k_{p,q}$ and $k'$ in the knot group $\pi K$ are conjugate. Hence they have the same images in $\pi K/\zeta\pi K$. Thus the order of twisting $m$ of a branched twist-spun knot is determined by the knot in all cases. (This was first proven in [6].)

If $d > 1$ then $\nu_d = Q(8) \times \mathbb{Z}/d\mathbb{Z}$ has an automorphism $\theta$ of order 6 such that $\nu_d \rtimes_\theta \mathbb{Z}$ is the group of a fibred 2-knot, but no self-homeomorphism of $S^3/\nu_d$ of order 6 has a fixed point. Hence $\nu_d$ is not the commutator subgroup of a twist spun knot. There are also fibred 2-knots with knot manifolds whose groups have centre $\mathbb{Z}^2$ but which do not satisfy condition (1) [9, page 318]. Thus this condition is not a consequence of the outer automorphism having finite order.

Let $N$ be a homology 3-sphere obtained by Dehn surgery on a knot $k$. Then $\nu = \pi_1(N)$ has weight one, and so does $\pi = \nu \times \mathbb{Z}$. Surgery on a loop in $N \times S^1$ representing a weight class gives a homotopy 4-sphere, and the cocore of the surgery is a 2-knot with group $\nu \times \mathbb{Z}$. If $k$ is a simple non-torus knot (such as $4_1$) then for all but finitely many Dehn surgeries $N$ is hyperbolic. The manifold $M = N \times S^1$ is then a 2-knot manifold which is a $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. However $\pi$ is not isomorphic to $\pi\tau_{m,s}k$ for any knot $k$, since $[\pi : \pi'.\zeta\pi] = 1$, and 1-twist spins are trivial. (This answers the question following [9, Corollary 17.11.1].)

In such cases the outer automorphism determined by the monodromy has finite order, but does not lift to an automorphism of finite order. Thus we cannot use Smith theory to show that the monodromy of the knot exterior has finite order, which is an essential hypotheses of [19, Proposition 6.1].

Condition (2) is unsatisfactory as part of an algebraic characterization of such groups. If $K$ is a 2-knot such that $\pi' = \pi K'$ is finitely presentable then $M(K)'$ is a $PD_3$-complex [9, Theorem 4.5]. Hence the indecomposable factors of $\pi'$ are either $PD_3$-groups or are virtually free. Consider the assertion

*$PD_3$-groups are fundamental groups of aspherical closed 3-manifolds.*

If this holds then a 2-knot group is the group of a fibred 2-knot if and only if $\pi'$ is finitely generated and $Q(8)$ is not a subgroup of any infinite indecomposable factor of $\pi'$ [10, Theorem 3.1]. We may then replace condition (2) by the condition

*$Q(8)$ is not a subgroup of any infinite indecomposable factor of $\pi'$.*

In particular, if $\pi'$ is indecomposable and infinite the conditions in Theorem 1 could be replaced by

*$\pi$ is a $PD_4$-group and conjugation by a weight element $t$ induces an automorphism of $\pi'$ of order $m$.*

We may also use [9, Theorem 4.5] to add a fifth characterization of knot groups with free commutator subgroup to [10, Theorem 2.1]: *If $\pi$ is a 2-knot group, $c.d.\pi = 2$ and $\pi'$ is finitely generated then $\pi'$ is free.* For if $M(K)'$ is a $PD_3$-complex and $c.d.\pi' \leqslant 2$ then $\pi'$ must be free. However we do not know whether "$\pi$ is a 2-knot group" is necessary (in the context of higher-dimensional knot groups).

**Theorem 4.** *The knot manifold $M(\tau_{m,s}k)$ has a geometric decomposition if and only if $k$ is a prime knot.*

*Proof.* A closed orientable 4-manifold $M$ with a geometric decomposition and $\chi(M) = 0$ is either a $\mathbb{S}^2 \times \mathbb{E}^2$- or $\mathbb{S}^3 \times \mathbb{E}^1$-manifold, or is aspherical [9, Theorem 7.2].

If $k$ is prime and $M_m(k)$ is covered by $S^3$ then $M(\tau_{m,s}k)$ is a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold. If $M_m(k)$ is aspherical then it has a JSJ decomposition which is equivariant with respect to the action of the monodromy. Since the (closed) monodromy has finite order the pieces of the decomposition of $M_m(k)$ give rise to a decomposition of $M(\tau_{m,s}k)$ with the corresponding product geometry $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$.

If $k$ is composite then $\pi'$ is a proper free product and $\pi_2(M_m(k)) \neq 0$. Hence $\pi\tau_{m,s}k$ is not virtually abelian and $M(\tau_{m,s}k)$ is not aspherical, so $M(\tau_{m,s}k)$ has no geometric decomposition. $\square$

It can be shown that finite volume $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds which are not closed are also $\widetilde{\mathbb{SL}}$-manifolds, and so proper pieces of type $\mathbb{H}^2 \times \mathbb{E}^2$ are also of type $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$.

**Corollary 5.** *If $K$ is a 2-knot with $\pi K \cong \pi\tau_{m,s}k$, where $k$ is a prime knot and $M_m(k)$ is aspherical, then $M(K)$ is s-cobordant to $M(\tau_{m,s}k)$.*

*Proof.* This follows from [9, Theorem 9.12] and [10, Theorem 3.4], since $Wh(\pi \times \mathbb{Z}) = 0$, as explained in [9, 9.§6]. $\square$

## 4. Uniqueness

In this section we shall summarize the results of Fukuda and Ishikawa, and complete their treatment of the torus knot case. We shall also show that a 2-knot group can be the group of only finitely many branched twist spins.

We begin with a simple argument for the torus-knot case.

**Theorem 6.** *Let $\pi$ be a 2-knot group such that $\zeta\pi \cong \mathbb{Z}^2$. Then $\pi$ is the group of at most finitely many branched twist spins.*

*Proof.* If $\pi = \pi\tau_{m,s}k$ then $k$ is a torus knot $k_{p,q}$, and $M(\tau_{m,s}k)$ is Seifert fibred over $S^2(m,p,q)$. Hence $\pi/\zeta\pi \cong \pi^{orb}(S^2(m,p,q))$, and so $\pi$ determines the triple $\{m,p,q\}$. Thus there are at most 3 possibilities for $m$. For each value of $m$ there are at most $\varphi(m)$ distinct branched twist spins. □

**Corollary 7.** *The torus knot $k_{p,q}$ is determined by $\pi$ and $m$.* □

Let $m,p,q > 1$ be pairwise relatively prime integers. Then $M(m,p,q)$ is a homology 3-sphere, and $M_m(k_{p,q}) \cong M_p(k_{m,q}) \cong M_q(k_{m,p}) \cong M(m,p,q)$. If $\nu = \pi_1(M(m,p,q))$ the the corresponding twist spins all have $\pi \cong \nu \times \mathbb{Z}$. Hence $\pi = \pi'.\zeta\pi$, and $m > [\pi : \pi'.\zeta\pi] = 1$. Thus the order of the twisting $m$ is not determined by the group. However, the order of the twisting must be one of $m,p$ or $q$. (Similarly, if $(m,q) = (p,q) = 1$ but $(m,p) > 1$ then the order of the twisting is $m$ or $q$. For example, $\pi\tau_4 k_{2,3} \cong \pi\tau_2 k_{3,4} \cong T_1^* \rtimes \mathbb{Z}$.)

Fukuda and Ishikawa give a stronger result for branched twist spins of the other prime knots. (They also use knot determinants to prove Corollary 7 above, with some restrictions on the parameters $p,q$.)

**Theorem** (Fukuda–Ishikawa [7]). *Let $k_1$ and $k_2$ be two prime knots, and suppose that $\pi\tau_{m,s}k_1 \cong \pi\tau_{m,s}k_2$ for some $m,s$ with $(m,s) = 1$. If $m = 2$ and $M_2(k_1)$ is Haken or if $m \geqslant 3$ and $k_1$ is not a torus knot then $k_1 \cong k_2$, and so $\tau_{m,s}k_1 \cong \tau_{m,s}k_2$.* □

The key points are that $k_1$ and $k_2$ are either hyperbolic or satellites, and the quotient $\pi/\zeta\pi$ is the orbifold fundamental group of 3-orbifolds corresponding to the branched coverings of $(S^3, k_1)$ and $(S^3, k_2)$. These orbifolds are hyperbolic or Haken, and therefore are determined by their fundamental groups. The argument of [7] shows in fact that if $(m, s_1) = (m, s_2) = 1$ and $\pi\tau_{m,s_1}k_1 \cong \pi\tau_{m,s_2}k_2$ then $k_1 \cong k_2$. However more may be needed to conclude that $\tau_{m,s_1}k_1 \cong \tau_{m,s_2}k_2$.

The next result is closely related to [17, Theorem 2.1].

**Theorem 8.** *Let $k$ be a knot and let $\theta$ be the automorphism of $\nu = \pi_1(M_m(k))$ induced by the canonical generator of the branched covering, for some $m > 1$. Let $s_1, s_2$ be integers such that $(m, s_1) = (m, s_2) = 1$. If $\tau_{m,s_1}k \cong \tau_{m,s_2}k$ then $\theta^{s_1}$ is conjugate in $Aut(\nu)$ to $\theta^{s_2}$ or $\theta^{-s_2}$.*

*Conversely, if* $\theta^{s_1}$ *is conjugate in* $Aut(\nu)$ *to* $\theta^{s_2}$ *or* $\theta^{-s_2}$ *then* $\pi\tau_{m,s_2}k \cong \pi\tau_{m,s_1}k$ *and the meridians of* $\tau_{m,s_1}k$ *and* $\tau_{m,s_2}k$ *are in the same weight orbit, up to inversion.*

*Proof.* The automorphism of $\nu$ induced by the canonical meridian for $\tau_{m,s}k$ is $\theta^s$. Let $t_1$ and $t_2$ be the canonical weight elements for $\tau_{m,s_1}k$ and $\tau_{m,s_2}k$. A homeomorphism of $S^4$ which carries $\tau_{m,s_1}k$ to $\tau_{m,s_2}k$ and preserves the homology class of the meridians induces an isomorphism $f_* : \pi\tau_{m,s_1}k \to \pi\tau_{m,s_2}k$ such that $f_*(t_1) = gt_2g^{-1}$, for some $g \in \nu$, and $f_*(\theta^{s_1}(x)) = g\theta^{s_2}(f_*(x))g^{-1}$, for all $x \in \nu$. Hence $d = c_g^{-1}f_*|_\nu$ is an automorphism of $\nu$ such that $d\theta^{s_1}d^{-1} = \theta^{s_2}$.

Conversely, if $d\theta^{s_1}d^{-1} = \theta^{s_2}$ for some automorphism $d$ of $\nu$ then setting $D(x) = d(x)$ for $x \in \nu$ and $d(t_1) = t_2$ defines an isomorphism $D : \pi\tau_{m,s_1}k_1 \to \pi\tau_{m,s_2}k_2$ which carries $t_1$ to $t_2$. Thus $D$ carries the weight orbit of $t_1$ onto the weight orbit of $t_2$. (Note that since $\theta^{s_1}$ and $\theta^{s_2}$ obviously commute, the criterion of §1 for comparison of the weight orbits reduces to requiring that $\theta^{s_1}$ be conjugate to $\theta^{s_2}$ in $Aut(\pi')$.)

The above arguments are easily adapted to the cases when we allow the homeomorphism carrying the image of one knot onto the other to reverse the homology class of the meridians. □

We know from Theorem 1 that if $k$ is prime then under the assumptions of the theorem $M(\tau_{m,s_1}k) \cong M(\tau_{m,s_2}k)$. There is a stronger result when $k$ is a torus knot and $\nu$ is not virtually solvable.

**Corollary 9.** *If* $k = k_{p,q}$ *where* $\frac{1}{m} + \frac{1}{p} + \frac{1}{q} < 1$ *then* $X(\tau_{m,s_1}k) \cong X(\tau_{m,s_2}k)$ *if and only if* $\theta^{s_1}$ *is conjugate in* $Aut(\nu)$ *to* $\theta^{s_2}$ *or* $\theta^{-s_2}$.

*Proof.* A homeomorphism of knot exteriors must carry meridians to meridians (up to inversion), and so the condition is necessary.

The hypothesis $\frac{1}{m}+\frac{1}{p}+\frac{1}{q} < 1$ is equivalent to $M_m(k)$ being aspherical and $\nu$ not solvable. Hence every automorphism of $\pi\tau_{m,s_2}k$ is realized by a self-homeomorphism of $M(\tau_{m,s_2}k)$ [11, Theorem 11.2.4]. Thus we may assume that there is a homeomorphism $h$ realizing an isomorphism $D : \pi\tau_{m,s_1}k_1 \to \pi\tau_{m,s_2}k_2$ which carries the (free) isotopy class of a meridian for $\tau_{m,s_1}k$ in $M(\tau_{m,s_1}k)$ to a meridian for $\tau_{m,s_2}k$ in $M(\tau_{m,s_2}k)$. Such a homeomorphism clearly restricts to a homeomorphism of the exteriors. The corollary then follows from the theorem. □

Note also that since torus knots are invertible, their (branched) twist spins are +amphicheiral [12]. Thus we may assume that $h$ is orientation-preserving.

TOP surgery may be used to give a similar result for $\tau_6k_{2,3}$ (corresponding to $\frac{1}{m} + \frac{1}{p} + \frac{1}{q} = 1$), and more generally when $M(K)$ is aspherical and $\pi$ is torsion-free and solvable. This is so if $K = \tau_3 4_1$ or $K = \tau_2 k$ for certain Montesinos knots $k$. (See [9, 16.§4].)

A further argument comes close to determining the number of distinct branched twist spins of torus knots.

**Theorem 10.** *Let $m, p, q, s, t$ be positive integers such that $(p, q) = (m, s) = (m, t) = 1$ and $\frac{1}{m} + \frac{1}{p} + \frac{1}{q} < 1$. If $\tau_{m,s}k_{p,q} \cong \tau_{m,t}k_{p,q}$ then $s^4 \equiv t^4$ mod $(m)$.*

*Proof.* We may assume that $m = 5$ or $m \geqslant 7$, for otherwise $s^4 = t^4 = 1$. Let $\nu = \pi_1(M(m, p, q))$. The base $B$ of the Seifert fibration of $M(m, p, q)$ is an orientable hyperbolic 2-orbifold, and $\pi^{orb}B = \nu/\zeta\nu$. The group of lifts of orientation preserving automorphisms of $B$ to $\widetilde{B} = \mathbb{H}^2$ is the normalizer of $\nu$ in $PSL(2, \mathbb{R})$. The characteristic automorphism $\theta$ corresponding to meridians of $\tau_m k_{p,q}$ acts orientably on $B$, and so determines an element of $PSL(2, \mathbb{R})$.

If $\tau_{m,s}k_{p,q} \cong \tau_{m,t}k_{p,q}$ then $\theta^t = \alpha\theta^{\pm s}\alpha^{-1}$ in $Aut(\nu)$. Hence $\theta^{t^2} = \alpha^2\theta^{s^2}\alpha^{-2}$, and $\alpha^2$ is orientation preserving. Let $\bar{g}$ and $\bar{h}$ be the images of $\theta$ and $\alpha^2$ in $PSL(2, \mathbb{R})$. Then $\bar{g}$ has order $m$, and $\bar{h}\bar{g}^{s^2}\bar{h}^{-1} = \bar{g}^{t^2}$. Since $\langle \bar{g}, \bar{h} \rangle$ is a subgroup of $PSL(2, \mathbb{R})$ and $m > 2$, this subgroup is either abelian or dihedral. Hence $s^2 \equiv \pm t^2$ *mod* $(m)$, and so $s^4 \equiv t^4$ *mod* $(m)$. ☐

If $m > 10$ then $\varphi(m) > 4$, and so there is an $r$ such that $(m, r) = 1$ and $r^4 \not\equiv 1$ *mod* $(m)$. Thus if $(m, t) = 1$ then $X(\tau_{m,rt}k_{p,q}) \not\cong X(\tau_{m,t}k_{p,q})$, although the knot groups are isomorphic.

**Corollary 11.** *If $m = r^k$ or $2r^k$ for some prime $r \equiv 3$ mod $(4)$ then $X(\tau_{m,s}k_{p,q}) \cong X(\tau_{m,t}k_{p,q})$ if and only if $s^2 \equiv t^2$ mod $(m)$.*

*Proof.* In this case $s^4 \equiv t^4$ *mod* $(m)$ if and only if $s \equiv \pm t$ *mod* $(m)$. ☐

In particular, if $m = 7$ then there are $3 = \frac{1}{2}\varphi(7)$ distinct knot exteriors $X(\tau_{7,s}k_{p,q})$, for each torus knot $k_{p,q}$.

If $k$ is composite then $M_m(k)$ is a proper connected sum, whereas if $k$ is prime then $M_m(k)$ is irreducible. Thus a branched twist spin of a prime knot is never a branched twist spin of a composite knot.

**Theorem 12.** *If $\pi$ is a group then $\pi \cong \pi\tau_{m,s}k$ for at most finitely many branched twist spins $\tau_{m,s}k$.*

*Proof.* Suppose that $\pi \cong \pi\tau_{m,s}k$ for some 1-knot $k$ and some $m, s$. Let $k = \Sigma_{i \leqslant r} k_i$ be the factorization of $k$ as a sum of prime knots. Then $\pi \cong G_1 *_{\mathbb{Z}} \cdots *_{\mathbb{Z}} G_r$ is the free product of the $G_i = \pi\tau_{m,s}k_i$, with amalgamation over subgroups generated by meridians, and $\pi' \cong \star_{i \leqslant r} G_i'$. Since $\pi'$ is finitely generated, it is a free product of indecomposable subgroups, and the factors which are not infinite cyclic are unique up to conjugacy and re-indexing. The subgroups $G_i'$ are indecomposable and none are infinite cyclic. Hence the set $\{G_i'\}$ and the number $r$ of prime factors of $k$ are determined by $\pi$.

The quotient of $\pi$ by the normal closure of the subgroups $\{G_j' : j \neq i\}$ is $G_i$. Then $G_i$ is the group of a branched twist simple knot in at most finitely many ways. This follows from [15] if $G_i'$ is finite. If $G_i'$ is infinite then $G_i' \cong \pi_1(N_i)$, where $N_i$ is aspherical and is Haken, hyperbolic or Seifert fibred. If $\zeta G_i' = 1$ then $m = [\pi : \pi'.\zeta\pi]$, by Corollary 3, and $k_i$ is not a torus knot. If $m > 2$ or if $m = 2$ and $N_i$ is Haken then $k_i$ is determined by $G_i$ and the pair $(m, s)$, by the work of Fukuda and Ishikawa [7] (cited above). If $m = 2$ and $N_i$ is hyperbolic then there are at most 9 possibilities for $k_i$ [20]. If $\zeta G_i \cong \mathbb{Z}^2$ then $k_i$ is a torus knot and there are at most 3 possibilities for $k_i$, by Theorem 6. Finally, if $\zeta G_i' \neq 1$ but $\zeta G_i \cong \mathbb{Z}$ then $m = 2$ and $k_i$ is a Montesinos knot; the number of possibilities for $k_i$ depends on $G_i'$ and may be large, but is finite.

Thus there are at most finitely many candidates for the summands $k_i$, and hence for $k$. Since there are only finitely many possibilities for $m$ (and hence $s$), by Corollary 3 and Theorem 6, the result follows. □

The following corollary follows immediately from Theorems 1 and 9.

**Corollary 13.** *If $K$ is a 2-knot and $\pi K'$ is indecomposable then $M(K) \cong M(\tau_{m,s}k)$ for at most finitely many branched twist spins $\tau_{m,s}k$.* □

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