THE $\mathbb{F}_2$-COHOMOLOGY RINGS OF 3-MANIFOLDS

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ABSTRACT. We give a new argument for the characterization of the cohomology rings of closed 3-manifolds with coefficients $\mathbb{F}_2$, first given by M. M. Postnikov (in terms of intersection rings) in 1948.

M. M. Postnikov characterized the $\mathbb{F}_2$-homology intersection rings of closed 3-manifolds in his first published paper [10], and D. Sullivan determined the $\mathbb{Z}$-cohomology rings of closed orientable 3-manifolds in [13]. The orientable case was settled comprehensively by V. G. Turaev, who considered not only cohomology with coefficients $\mathbb{Z}/n\mathbb{Z}$ for all $n \geqslant 0$, but determined the interactions of these cohomology rings with each other and with the torsion linking pairing [14].

Postnikov showed that the $\mathbb{F}_2$-homology intersection ring of a closed 3-manifold M is a finite graded $\mathbb{F}_2$-algebra which satisfies 3-dimensional Poincaré duality and the"Postnikov-Wu identity" with respect to a distinguished element $w = w_1(M)$, and conversely every such "MS-algebra" is isomorphic to such a cohomology ring. (The terms in quotation marks are defined in §5 below.) The constructive part of his argument used induction on the rank of the degree-1 component of the algebra, with bases corresponding to S^3 and $S^2\tilde{\times}S^1$, for the cases $w = 0$ and $w \neq 0$, respectively. The inductive step in [10] used (in today's terminology) surgery on a knot K in a 3-manifold M such that K has trivial image in $H_1(M;\mathbb{F}_2)$.

We review briefly the results of Sullivan and Turaev in §2 and §3. The main body of the paper is in §4–§9, where we give a new argument for Postnikov's result. The basic idea (superposition of elementary models) is quite simple, but its application to nonorientable 3-manifolds involves some effort. Although we shall use the language of cohomology, our calculations rest largely upon the intersections of curves and surfaces in a 3-manifold, as in Postnikov's original account. Our contributions are merely to give a direct link presentation of a suitable 3-manifold, suppressing the induction, and to make more explicit the penultimate paragraph of Postnikov's argument, which consists of three sentences, beginning with "*Okazivayetsya, shto* ...", meaning roughly "It turns out that ...".

When the homology is torsion free the $\mathbb{F}_p$-cohomology ring is determined by the integral cohomology ring. In the final two sections we show that this is not always so when there is torsion.

1. NOTATION

If A is a finitely generated abelian group let tA be its torsion subgroup, and let $A^* = Hom(A,\mathbb{Z})$ and $A^{*n} = Hom(A,\mathbb{Z}/n\mathbb{Z})$, for each $n > 1$. If R is another abelian group then there is a canonical epimorphism from $Hom(A,R)$ onto $Hom(tA,R)$, with kernel $Hom(A/tA,R)$.

2020 *Mathematics Subject Classification.* 57M45.

Key words and phrases. 3-manifold, cohomology ring, *mod*-(2) coefficients.

If R is a commutative ring and $\{e_1, \dots, e_d\}$ is a basis for a free R-module V then the Kronecker dual basis for $V^* = Hom_R(V, R)$ is the basis $\{e_1^*, \dots, e_d^*\}$ determined by $e_i^*(e_i) = 1$ and $e_i^*(e_j) = 0$ for all $j \neq i$ and all i.

If G is a group let G' and ζG be the commutator subgroup and centre of G, and let $G^{ab} = G/G'$ be its abelianization. Let $I(G)$ be the preimage in G of tG^{ab}. Let $X^r(G)$ be the verbal subgroup generated by all rth powers g^r with $g \in G$.

If L is an m-component link let L_i be its ith component, and let $M(L; \mathcal{F})$ be the closed 3-manifold obtained by $\mathcal{F}$-framed surgery on L. If each component has the 0-framing we shall write just $M(L)$, and if each component has framing with slope p we shall write $M(L; p)$. Spanning surfaces for link components and closed surfaces representing Poincaré duals of classes in $H^1(M; \mathbb{F}_2)$ may be nonorientable, and we shall not comment further on this possibility.

Let $L_{2,4}$ and Bo be the (2,4)-torus link and the Borromean rings 3-component link. (These are the links 4_1^2 and 6_2^3 in Rolfsen's tables [11].) Let $Bo(n)$ be the link obtained by replacing Bo_3 of Bo by its $(1, n)$-cable.

If $A \in GL(2, \mathbb{Z})$ then $MT(A)$ is the mapping torus of the self-homeomorphism of the torus T induced by A.

2. CHARACTERISTIC NOT 2

Let M be a closed 3-manifold and let $\pi = \pi_1(M)$. Let $R = \mathbb{Z}$, a prime field $\mathbb{Q}$ or $\mathbb{F}_p$, with p odd. Let $H = \pi^{ab}$ and let $H^* = Hom(H, R)$. If M is orientable then $H_3(M; R) \cong R$, and cap product with a fundamental class $[M]$ defines Poincaré duality isomorphisms $D_2 : H^2(M; R) \to R \otimes H$ and $D_3 : H^3(M; R) \to R$, while cup product and duality define homomorphisms $\gamma : \wedge_2 H^* \to H^2(M; R)$ and $\mu = \mu_M : \wedge_3 H^* \to R$. (Alternating trilinear functions such as μ may be identified with elements of $\wedge_3 H$.) These satisfy the equations

$$D_3(a \smile D_2^{-1}(h)) = a(h) \quad \forall\ a \in H^* \text{ and } h \in H$$

and

$$\mu(a \wedge b \wedge c) = a(D_2 \gamma(b \wedge c)) \quad \forall\ a, b, c \in H^*.$$

If R is a prime field, (with characteristic $\neq 2$) or if $R = \mathbb{Z}$ and H is torsion-free then the cohomology ring $H^*(M; R)$ is determined by H, μ and the duality isomorphisms D_2 and D_3, via these equations. (More explicitly, let $\mathcal{H}^0 = R$, $\mathcal{H}^1 = H^*$, $\mathcal{H}^2 = H$ and $\mathcal{H}^3 = R\varepsilon_3$, and let rs be the unique solution of "$t(rs) = \mu(r \wedge s \wedge t)\varepsilon_3\ \forall\ t \in R^1$", and $rh = r(h)\varepsilon_3$, for $r, s \in \mathcal{H}^1$ and $h \in \mathcal{H}^2$. Then $rst = \mu(r \wedge s \wedge t)\varepsilon_3$, for all $r, s, t \in H^*$, and $\mathcal{H}^*(H, \mu) = \oplus_{i=0}^3 \mathcal{H}^i$ is a graded ring. We may use D_2 and D_3 to determine an isomorphism $H^*(M; R) \cong \mathcal{H}^*(H, \mu)$.) If $H^* \neq 0$ then D_2 determines D_3, by the first equation above.

Theorem (Sullivan [13]). *If $R = \mathbb{Z}$ then every such pair (H, μ) is realizable by some closed orientable 3-manifold with torsion-free homology.* □

We may assume that the 3-manifold is irreducible [9].

If M is not orientable and F is a field of characteristic $\neq 2$ hen $H^3(M; F) = 0$, and so $\beta_2(M; F) = \beta_1(M; F) - 1$. Up to automorphisms of $H^1(M; F)$ and $H^2(M; F)$, the only issue of interest is the rank of c_p. If $P_c = \#^r \mathbb{RP}^2$ and $M = S^1 \times P_c$ then c_p is an epimorphism for all characteristics $p \neq 2$. Hence connected sums of $S^1 \times P_c$ with copies of $S^1 \times S^2$ realize all possible combinations of rank and Betti numbers satisfying $rk(c_p) \leqslant \beta_2(M; \mathbb{F}) - 1$.

When $R = \mathbb{Z}$ or $\mathbb{F}_2$ the Poincaré duals of Kronecker duals of a basis for $H_1(M;R)$ represented by simple closed curves in M may be represented by closed surfaces in M which meet one such curve transversely in one point and are disjoint from the other curves. However these closed surfaces are generally not pairwise disjoint, and we may use their intersections to identify cup products. See [5].

3. INTEGER COEFFICIENTS: TORSION

When $R = \mathbb{Z}$ and $H_1(M;\mathbb{Z})$ has nontrivial torsion then γ is no longer determined by μ_M and the above equation. The torsion subgroup has a complementary direct summand, but the splitting is not canonical. Cup products with coefficients in other rings and their compatibilities with integral cup product must also be considered. V.G.Turaev gave a definitive account in [14].

For each $n > 1$ let $\nu_n = \nu_{nM} : \wedge_3 H^1(M;\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ be defined by

$$\nu_n(X,Y,Z) = (X \cup Y \cup Z) \cap [M], \quad \text{for all } X,Y,Z \in H^1(M;\mathbb{Z}/n\mathbb{Z}).$$

Then ν_{nM} and Poincaré duality together determine the ring $H^*(M;\mathbb{Z}/n\mathbb{Z})$. Every 3-form $\nu : \wedge_3(\mathbb{Z}/n\mathbb{Z})^\beta \to \mathbb{Z}/n\mathbb{Z}$ lifts to a 3-form $\widehat{\nu} : \wedge_3\mathbb{Z}^\beta \to \mathbb{Z}$. Hence it is an immediate consequence of Sullivan's construction that every such 3-form ν can be realized as ν_{nM} for some closed orientable 3-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^\beta$. If p is an odd prime it follows that every finite graded-commutative graded $\mathbb{F}_p$-algebra satisfying 3-dimensional Poincaré duality is the $\mathbb{F}_p$ cohomology ring of such a 3-manifold.

The Bockstein homomorphism $\beta_{\mathbb{Q}/\mathbb{Z}} : Hom(H,\mathbb{Q}/\mathbb{Z}) \to H^2(M;\mathbb{Z})$ has image $Ext(H,\mathbb{Z})$, the torsion subgroup of $H^2(M;\mathbb{Z})$. We may use this to define the torsion linking pairing $\ell : tH \times tH \to \mathbb{Q}/\mathbb{Z}$ by

$$\ell(u,v) = (D_2 \circ \beta_{\mathbb{Q}/\mathbb{Z}})^{-1}(v)(u) \quad \forall\ u,v \in tH.$$

This pairing is nonsingular and symmetric, and ℓ and $\beta_{\mathbb{Q}/\mathbb{Z}}$ determine each other (given D_2). Every such pairing is realizable by some $\mathbb{Q}$-homology 3-sphere [6, Theorem 6.1]. Taking connected sums shows that every such triple (H,μ,ℓ) is realizable by some 3-manifold. On the other hand, μ and ℓ are independent invariants.

It is easily verified that if $x \in H^1(M;\mathbb{Z}/n\mathbb{Z})$ then $x^2 = \frac{n}{2}\beta_{\mathbb{Q}/\mathbb{Z}}(x)$. (It suffices to check this for $x = id_{\mathbb{Z}/n\mathbb{Z}}$, considered as an element of $H^1(\mathbb{Z}/n\mathbb{Z};\mathbb{Q}/\mathbb{Z}) = H^1(\mathbb{Z}/n\mathbb{Z};\mathbb{Z}/n\mathbb{Z})$.) Let $\psi_n : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ be the standard inclusion. If $x \in H^{*n}$ let $\hat{x}$ be the element of tH such that $\ell(\hat{x},a) = \psi_n(x(a))$ for all $a \in tH$. Turaev showed that

$$\psi_n(\nu_n(x,x,y)) = \frac{n}{2}\ell(\hat{x},\hat{y}) \text{ for all } x,y \in H^{*n}.$$

When $n = 2$ this condition implies the orientable case of the Postnikov-Wu identity invoked below. (Note that if n is odd then both sides are 0.)

Let A be a finitely generated abelian group, let $\ell : tA \times tA \to \mathbb{Q}/\mathbb{Z}$ be a nonsingular symmetric pairing, and suppose that $\nu : \wedge_3 A^* \to \mathbb{Z}$ and $\nu_n : \wedge_3 A^{*n} \to \mathbb{Z}/n\mathbb{Z}$ is a system of alternating trilinear functions which are compatible under reduction *mod-n*. Then Turaev showed that such a group A, pairing ℓ and system of trilinear functions may be realizable by the homology and cohomology of a closed orientable 3-manifold if and only if the above condition deriving from the interaction of the Bockstein $\beta_{\mathbb{Q}/\mathbb{Z}}$ with the cup-square holds for all n dividing the order of tA [14].

4. SURGERY ON LINKS

In the next six sections we shall give an alternate approach to Postnikov's result on the $\mathbb{F}_2$-cohomology rings of 3-manifolds. Our examples shall be constructed by surgery on framed links. Related constructions were used by Turaev and Postnikov, although in the latter case "surgery" was not yet a mathematical term. (Sullivan uses instead Heegaard decompositions.)

In the orientable case, in order to construct a closed 3-manifold with M with $\beta_1(M;\mathbb{F}_2)=\rho$ we modify various of the components of a trivial ρ-component link ρU in S^3, using local moves which involve replacing trivial 2- or 3-component tangles in a ball by other tangles from a limited repertoire.

The components of the tangles in Figure 1 represent distinct link components. Move (a) changes the linking number by ± 2, and an application of this move changes the 2-component trivial link $2U$ into $L_{2,4}$. Move (b) does not changing the linking number. An application of this move changes the 3-component trivial link $3U$ into Bo. Moves (a) and (b) (in either order) change $3U$ into the link of Figure 3. These moves do not change the knot types of the link components.

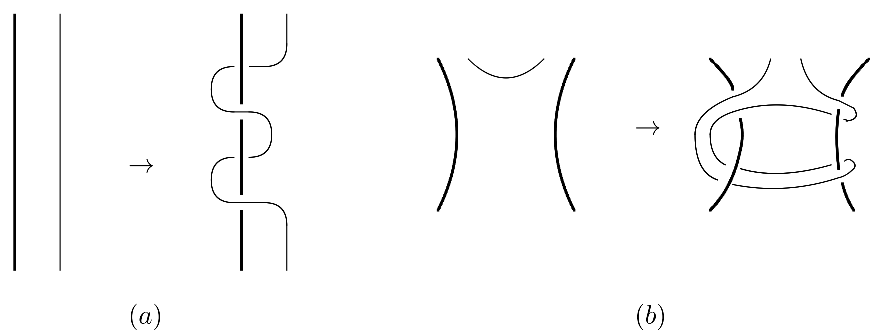

Figure 1. Two tangle moves.

In each of these cases a given set of spanning surfaces for the components of the original link may be modified within the ball containing the tangle without introducing new intersections outside the ball. (For instance, applying move (b) to the trivial 3-component link $3U$ gives Bo, as in Figure 2.)

In the nonorientable case we start with nontrivial links in $S^2\tilde{\times}S^1$ which are disjoint unions of several basic 2- or 3-component links with all components orientation preserving. (Each component K then has "longitudes" which bound surfaces in $S^2\tilde{\times}S^1$.) We may assume that each component meets a fixed copy of the fibre S^2 transversely, and shall make our modifications to trivial 2- or 3-component tangles in balls lying in the complementary region $S^2\times(0,1)$.

Lemma 1. *Let $M=M(Bo(n))$, and let a,b,c be the basis for $H_1(M;\mathbb{Z})$ determined by the meridians of $Bo(n)$. Let a^*,b^*,c^* be the Kronecker dual basis for $H^1(M;\mathbb{Z})$. Then $(a^*\cup b^*\cup c^*)\cap[M]=n$.*

Proof. This is most easily seen using a mixture of algebra and geometry. The first two components bound disjoint discs in their mutual complement. Each of these discs meets the third component in 2 points. If we delete neighbourhoods of these intersection points and attach tubes which surround arcs of the third component, we obtain punctured tori in the exterior of $Bo(n)$, which may closed off in M by copies of the surgery discs used in forming M. The resulting tori represent the Poincaré duals of a^* and b^*, and intersect along a meridian for the third component of Bo, as in Figure 2. The latter meridian is homologous in M to $n.c$, since the third component of $Bo(n)$ is the $(1,n)$-cable of the third component of Bo. Hence $(a^* \cup b^* \cup c^*) \cap [M] = n$. □

If H has basis $\{e_1, \ldots, e_\beta\}$ then the simple 3-forms $e_{ijk} = e_i \wedge e_j \wedge e_k$ with $i < j < k$ form a basis for $\wedge_3 H$.

Theorem 2. *Let $M = M(L)$, where L is an m-component link with ordered and oriented components and such that each 3-component sublink is either trivial or is a copy of Bo. Let I be the set of ordered triples $i < j < k$ such that $L_i \cup L_j \cup L_k$ is isotopic to Bo, and let $\{a_i\}$ be the meridianal basis of $H_1(M;\mathbb{Z})$. Then*

$$\mu_M = \Sigma_{(i,j,k)\in I} a_i \wedge a_j \wedge a_k.$$

Proof. Let $e_1, \ldots, e_m$ be the basis for $H_1(M;\mathbb{Z})$ determined by the meridians for L, and let $e_1^*, \ldots, e_m^*$ be the Kronecker dual basis for $H^1(M;\mathbb{Z})$. To calculate $\mu(e_i^*, e_j^*, e_k^*)$ it suffices to use Poincaré duals for the e_i^* as in Lemma 1. □

Figure 2 shows two punctured tori spanning components Bo_1 and Bo_2 of Bo, with intersection a meridian c for the third component.

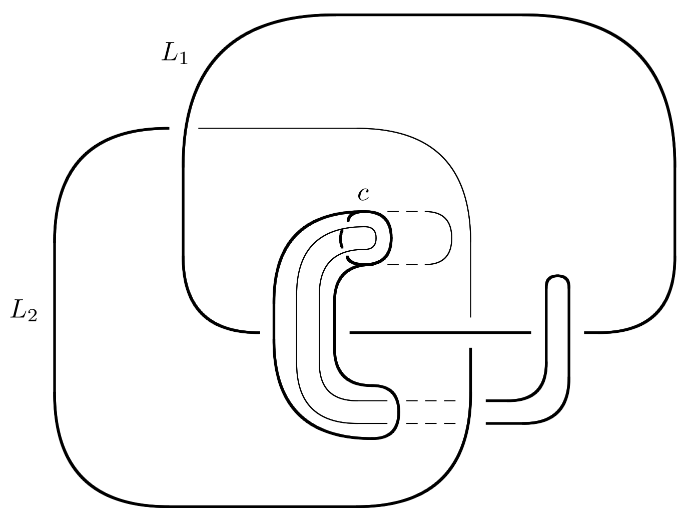

Figure 2.

Lemma 1 and Theorem 2 suffice to recover Sullivan's result for $H \cong \mathbb{Z}^\beta$ with $\beta \leqslant 5$. If $\beta < 3$ then $\mu = 0$ and if $\beta = 3$ then $\mu = re_1^* \wedge e_2^* \wedge e_3^*$, for some $r \in \mathbb{Z}$. If $\beta = 4$ then 3-forms are dual to 1-forms, while if $\beta = 5$ then 3-forms are dual to alternating 2-forms, and so μ is equivalent (under the action of $GL(\beta,\mathbb{Z})$) to $re_1^* \wedge e_2^* \wedge e_3^*$ or $re_1^* \wedge e_2^* \wedge e_3^* + se_1^* \wedge e_4^* \wedge e_5^*$ (respectively), for some $r, s \in \mathbb{Z}$.

If H^{*p} has basis $\{e_1, \ldots, e_\gamma\}$ then the simple 3-forms $e_{ijk} = e_i \wedge e_j \wedge e_k$ with $i < j < k$ form a basis for $\wedge_3 H^{*p}$. If $\nu = \Sigma \eta_{ijk} e_{ijk}$, where each $\eta_{ijk} = 0$ or 1 then we may realize ν by a closed orientable 3-manifold with $H_1(M;\mathbb{Z}) \cong \mathbb{F}_p^\gamma$.

A similar argument applies for $M(Bo(n);2)$ and coefficients $\mathbb{F}_2$ (and probably also to $M(Bo(n);p)$ and coefficients $\mathbb{F}_p$), but we may need to extend intersection theory to $\mathbb{Z}/p\mathbb{Z}$-manifolds [12], or alternatively use a more algebraic argument, working modulo the cores $S^1 \times \{0\}$ of the surgeries.

This construction could then be extended to the case $H \cong \mathbb{Z}^\beta \oplus \mathbb{F}_p^\rho$, by using 0-framed surgeries on the first β components. However another idea seems necessary to pick up coefficients other than 0 or 1.

If F is a field and $\beta \leqslant 7$ then every 3-form on F^β is equivalent under the action of $GL(\beta,F)$ to a standard 3-form of the above type. This is clear if $\beta \leqslant 3$, and also if $\beta = 4$, for then 3-forms are dual to 1-forms. If $\beta = 5$ then 3-forms are dual to 2-forms, and so correspond to skew-symmetric pairings. Hence there are just two equivalence classes of nonzero forms, represented by e_{123} and $e_{123}+e_{145}$. The case $\beta = 6$ involves more work, but there are just two more standard forms, represented by $e_{123}+e_{456}$ and $e_{162}+e_{243}+e_{135}$. If $\beta = 7$ there are at most 12 equivalence classes, some involving coefficients other than 0 or 1, but the above construction still suffices. See [1] for details of the standard forms in this case. On the other hand, if $\beta > 8$ then $\dim_F GL(\beta,F) = \beta^2 < \dim_F \wedge_3 F^\beta = \binom{\beta}{3}$, and the number of equivalence classes is unbounded as the order of F increases. The result of Turaev shows that we should not need to appeal to [1].

5. CHARACTERISTIC 2

In the strictly antisymmetric cases (characteristic 0 or odd), there is only one possible nonzero cup product involving three given degree-1 classes. When $R = \mathbb{F}_2$ the homomorphisms γ and μ must be replaced by homomorphisms from the symmetric products $\odot^2 H^* \to H^2(M;\mathbb{F}_2)$ and $\odot^3 H^* \to H^3(M;\mathbb{F}_2)$. There are many more possibilities for nonzero triple products, and we should now consider also nonorientable 3-manifolds. Postnikov gave a complete account of the $\mathbb{F}_2$-intersection rings of 3-manifolds. He did not assume orientability, and his description of $H^*(M;\mathbb{F}_2)$ includes also the relation $uvw = u^2v + uv^2$, for all $u, v \in H^1(M;\mathbb{F}_2)$ [10]. (Here w is the orientation character.) This may now be seen as an application of the Wu relation $Sq^1 z = wz$, for all $z \in H^{n-1}(M^n;\mathbb{F}_2)$, which first appeared later [16].

We have adapted Postnikov's terminology to the cohomological formulation. An *MS-algebra* is a finite commutative graded $\mathbb{F}_2$-algebra $\mathcal{A}^* = \oplus_{i=0}^3 \mathcal{A}^i$ such that $\dim \mathcal{A}^0 = \dim \mathcal{A}^3 = 1$, $\mathcal{A}^j = 0$ for $j > 3$, multiplication defines a perfect pairing from $\mathcal{A}^1 \times \mathcal{A}^2$ into $\mathcal{A}^3$, so that $\mathcal{A}^*$ satisfies formal Poincaré duality of dimension 3, and which has a distinguished element $w \in \mathcal{A}^1$ such that the *Postnikov-Wu identity*

$$wxy = x^2y + xy^2$$

holds for all $x, y \in \mathcal{A}^1$. An *MS-algebra isomorphism* is a ring isomorphism under which the distinguished elements w correspond, and $\mathcal{A}^*$ is *orientable* if $w = 0$. We shall abbreviate "by the nonsingularity of multiplication from $\mathcal{A}^1 \times \mathcal{A}^2$ to $\mathcal{A}^3$" to "by nonsingularity".

In the orientable case the Postnikov-Wu identity is an easy consequence of standard facts about the reduced Bockstein homomorphisms $\overline{\beta}_2$. If X is an orientable PD_3-complex then reduction mod (2) maps $H^2(X;\mathbb{Z})$ onto $H^2(X;\mathbb{F}_2)$, since $H^3(X;\mathbb{Z}) \cong \mathbb{Z}$. Hence $\overline{\beta}_2$ is trivial on $H^2(X;\mathbb{F}_2)$, and so $u^2v + uv^2 = Sq^1(u)v + uSq^1(v) = \overline{\beta}_2(uv) = 0$ for all $u, v \in H^1(X;\mathbb{F}_2)$.

6. THE ORIENTABLE CASE

In this section we shall consider the orientable case, which is somewhat easier, as we may realize a basis for $\mathcal{A}^1$ by meridians for a suitable link in S^3. The Postnikov-Wu identity also simplifies to $x^2y = xy^2$ for all $x, y \in \mathcal{A}^1$. We shall design a framed link representing the 3-manifold, guided by the nonzero triple products. The basic ingredients are $L_{2,4}$, Bo and the link of Figure 3 below.

Let K be a knot in S^3 with exterior $X(K)$, meridian μ_K and longitude λ_K. A p-framed surgery on K is determined by a homeomorphism $\phi : \partial X(K) \to S^1 \times D^2$ such that $\phi(\lambda_K + p\mu_K) = \partial D^2$. After composition with a self-homeomorphism of $S^1 \times D^2$, if necessary, we may assume that $\phi(\mu_K) = S = S^1 \times \{*\}$. It then follows that $\phi(\lambda_K)$ is a simple closed curve C_p representing the homology class $[\partial D^2] - p[S]$ in $H_1(S^1 \times \partial D^2)$. If $p = 0$ this curve clearly bounds a copy of D^2; if $p = \pm 2$ then it bounds a ribbon with a half-twist, i.e., a copy of the Möbius band Mb. [In general, C_p bounds a $\mathbb{Z}/p\mathbb{Z}$-manifold. If p is even is there a natural desingularization?]

Let $M = M(L; \mathcal{F})$, where L is an m-component link in which all pairwise linking numbers are even and in which each component has even framing. The meridians for the link components represent a canonical basis for $H_1(M; \mathbb{F}_2)$, and the Kronecker duals of this basis give a basis $\{x_1, \dots, x_m\}$ for $H^1(M; \mathbb{F}_2)$. Let F_i be a Seifert surface for the the i-th component L_i in S^3. The Poincaré dual of x_i in $H_2(M; \mathbb{F}_2)$ is represented by the surface $\widehat{F}_i$ which is the union of F_i with a spanning surface for the longitude λ_{L_i} in $S^1 \times D^2$. If all the linking numbers are 0 or ± 2 then after attaching handles to F_i if necessary, we may assume that it does not meet any other component of L.

If all linking numbers are 0 then we may assume that F_i is orientable, and then x_i^2 and x_i^3 are supported by the cocore of the surgery. In this case $x_i^2 = 0$ if L_i has framing 0, while $x_i^3 \neq 0$ if the framing is ± 2. Moreover the triple products $x_i^2 x_j$ and $x_i x_j x_k$ then depend only on the sublinks $L_i \cup L_j$ and $L_i \cup L_j \cup L_k$ involved and the framings of these components. (If the framings and linking numbers are even but some are not 0 or ± 2 then the components bounds immersed spanning surfaces which are disjoint from the other components. Our approach might extend to these cases, but we do not know whether the intersection theory applies adequately for transversely immersed surfaces.)

The (2,4)-torus link is the simplest 2-component link with linking number 2. Its link group has the presentation $\langle a, b \mid bab^{-1}a = ab^{-1}ab \rangle$. The longitudes are $\ell_a = aba^{-1}b$ and $\ell_b = bab^{-1}a$, and 0-framed surgery gives the quaternionic 3-manifold $S^3/Q(8)$.

Let u and v be the Kronecker duals of the images of the meridians a and b in $H_1(S^3/Q(8); \mathbb{F}_2)$. Then u^2 and v^2 are nonzero, since $Q(8)^{ab}$ has exponent 2. On the other hand, $u^3 = v^3 = 0$, and so $u^2 \neq v^2$ and $u^2 v = uv^2 \neq 0$, by nonsingularity. In this case the kernel of cup product is generated by $u \odot u + u \odot v + v \odot v$. These are standard facts about $H^*(Q(8); \mathbb{F}_2)$, but can be explained in our terms as follows. The Poincaré dual of u is represented by a Klein bottle $\widehat{F}$ in $S^3/Q(8)$. The intersection of two transverse copies of $\widehat{F}$ is a meridian for the other component, and can be pushed off $\widehat{F}$. Hence the intersection of three mutually transverse copies of $\widehat{F}$ is empty.

If L is the link of Figure 3 then $M(L)$ is the "half-turn" flat 3-manifold $MT(-I_2)$ with $\pi_1 MT(-I_2) \cong \mathbb{Z}^2 \rtimes_{-I} \mathbb{Z}$. Poincaré duality and consideration of subgroups and quotient groups shows that

$$H^*(MT(-I_2);\mathbb{F}_2) \cong \mathbb{F}_2[t,u,v]/(t^2,u^3,v^3,tu^2,tv^2,tuv+u^2v,u^2v+uv^2).$$

Realization: Let $\mathcal{A}^*$ be an orientable MS-algebra and let $\rho = \dim_{\mathbb{F}_2}\mathcal{A}^1$. Then $u^2v = uv^2$ and so $(u+v)^3 = u^3+v^3$, for all $u,v \in \mathcal{A}^1$. Let $\nu : \odot^3\mathcal{A}^1 \to \mathcal{A}^3$ be the triple product. If $\{x_1,\dots,x_\rho\}$ is a basis for $\mathcal{A}^1$, we let $\nu_{ijk} = \nu(x_i \odot x_j \odot x_k)$ for $1 \leqslant i,j,k \leqslant \rho$. Given $u,v,z \in \mathcal{A}^1$, there are 10 possible triple products involving just these elements, but their values are constrained by the Postnikov-Wu identity, and the number of possibilities to consider may be reduced further by judicious choice of basis.

If $x^2 = 0$ for all $x \in \mathcal{A}^1$ then we may model $\mathcal{A}^*$ by $M(L)$, where L is a ρ-component link with all nontrivial 3-component sublinks copies of Bo.

Suppose next that $Sq^1x_i = x_i^2 \neq 0$ for $i \leqslant \sigma$, for some $\sigma > 0$. We may assume that $\{x_{\sigma+1},\dots,x_\rho\}$ spans $\mathrm{Ker}(Sq^1)$, and so Sq^1 maps $X = \langle x_1,\dots,x_\sigma\rangle$ bijectively to $Sq^1X = \langle x_1^2,\dots,x_\sigma^2\rangle$. If $x^3 = 0$ for all $x \in \mathcal{A}^1$ then after a change of basis, if necessary, we may assume that the restricted pairing is block diagonal, with diagonal blocks $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$. For if $x_1^2y \neq 0$ for some $y \in \mathcal{A}^1$ then $x_1y^2 \neq 0$, and we may set $x_2 = y$. We then modify the other basis elements x_j with $j > 2$ within their cosets *mod* $\langle x_1,x_2\rangle$ so that $x_1^2x_j = x_2^2x_j = 0$. We continue by induction on k to modify the x_{2j-1} and x_{2j} with $j > k$ so that $x_{2j-1}^2x_{2j} \neq 0$ and $x_i^2x_{2j-1} = x_i^2x_{2j} = 0$ for all $i \leqslant 2k$. In particular, σ is even.

We start with a ρ-component link which splits as a union of $\frac{\sigma}{2}$ copies of $L_{2,4}$ and a trivial link with $\rho-\sigma$ components, and construct the desired link L by modifying some trivial 3-tangles, as in Figure 1.(b). The generators of $\mathcal{A}^1$ correspond to the Kronecker duals of the meridians of L. If $\nu_{ijk} \neq 0$ for some $i<j<k$ and $x_i^2x_j = x_i^2x_k = x_j^2x_k = 0$ then we arrange for $L_i \cup L_j \cup L_k$ to be a copy of Bo. However if (say) $x_i^2x_j \neq 0$ then we use instead the link of Figure 3, in which the components L_i and L_j are linked. We give all components framing 0.

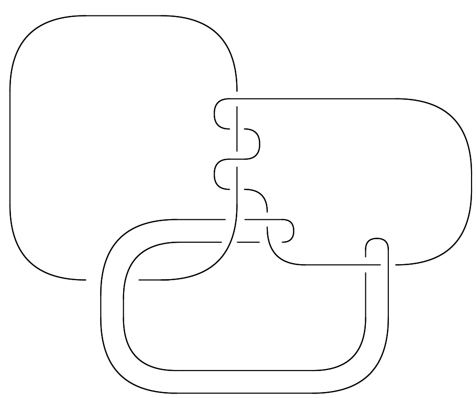

Figure 3. Linking numbers 0, 0, 2.

If there is an x such that $x^3 \neq 0$ then we may assume that $x_1^3 \neq 0$. After replacing x_i by $x_i + x_1$, if necessary, we may assume that $x_1^2x_i = 0$ for all $i > 1$. Hence the only possible nonzero triple products involving x_1 are of the form xyz with x,y,z linearly independent. In order to achieve this we choose our link so that all linking numbers $\ell(L_1, L_i)$ with $i > 1$ are 0, and the framing of L_1 is 2. In this case $\mathrm{Ker}(Sq^1)$ has odd codimension. The construction proceeds as before. (With the above choices the multiplication pairing between X and Sq^1X is again block diagonal, but there may be several basis elements x with $x^3 \neq 0$, and so the diagonal blocks are $[1]$ and $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$.)

Examples. If M is orientable and $\rho = \beta_1(M;\mathbb{F}_2) = 1$ then $H^*(M;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^4)$ or $\mathbb{F}_2[x,u]/(x^2,u^2)$, with $x \in H^1(M;\mathbb{F}_2)$ and $u \in H^2(M;\mathbb{F}_2)$. The simplest examples are $\mathbb{RP}^3 = L(2,1)$, $S^1 \times S^2$ and $L(4,1)$.

It follows from the Postnikov-Wu identity that $x^3 + y^3 = (x+y)^3$ for all x, y of degree 1. Hence if $\{x,y\}$ is a basis for $H^1(M;\mathbb{F}_2)$ such that $x^3 \neq 0$ then we may assume that $y^3 \neq 0$ also. Nonsingularity of Poincaré duality then implies that $xy = 0$, and so $x \odot y$ generates the kernel of cup product. For example, $\mathbb{RP}^3\#\mathbb{RP}^3$. Otherwise, if $\{x,y\}$ is a basis for $H^1(M;\mathbb{F}_2)$ and $z^3 = 0$ for all z of degree 1 then $x^2y = xy^2 \neq 0$, and the kernel of cup product is generated by $x\odot x + x\odot y + y \odot y$, as for the quaternion manifold $S^3/Q(8)$.

Two more 3-component links shall play a role in the nonorientable case.

Let L be the link of Figure 4, and let A, P and X be the classes in $H_1(M(L);\mathbb{Z})$ represented by the meridians a, p and x. Then $2(A+P) = 2X = 0$. Let $\{u,v,z\}$ be the basis for $H^1(M(L);\mathbb{F}_2)$ which is Kronecker dual to the basis for $H_1(M(L);\mathbb{F}_2)$ represented by the meridians (so that $u = a^*$, $v = p^*$ and $z = x^*$). Then $u^3 = v^3 = z^3 = 0$ (since the components have framing 0), while $z^2 \neq 0$ (since $2X = 0$). Let $\langle\langle x\rangle\rangle$ be the normal closure of the image of x in $\pi_1M(L)$. Then $G = \pi_1M(L)/\langle\langle x\rangle\rangle$ has presentation $\langle a, p \mid (ap^{-1})^2 = 1\rangle$. Then $u^2 = uv = v^2$, since these relations hold in $H^*(G;\mathbb{F}_2)$, and so $u^2v = uv^2 = u^3 = 0$. Hence $uz^2 = vz^2 = uvz \neq 0$, by nonsingularity, and so

$$H^*(M(L);\mathbb{F}_2) \cong \mathbb{F}_2[u,v,z]/(u^2+uv, v^2+uv, u^2v, u^2z+uz^2, v^2z+vz^2, z^3).$$

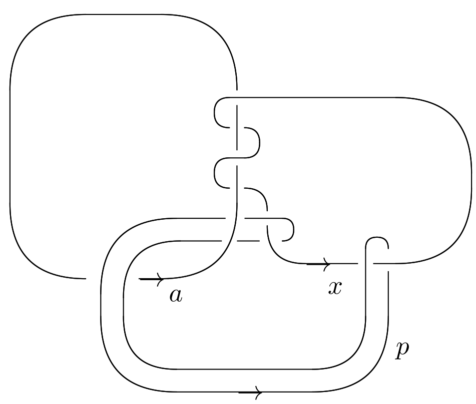

Figure 4. Linking numbers 0, 2, 2.

The link of Figure 5 has a 3-fold symmetry (about the axis through $\bullet$) which permutes the components. If A, P and X are the classes in $H_1(M(L);\mathbb{Z})$ represented by the meridians then $2A+2P = 2A+2X = 2P+2X = 0$. Hence $H_1(M(L);\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Let $\{u,v,z\}$ be the basis for $H^1(M(L);\mathbb{F}_2)$ which is Kronecker dual to the basis for $H_1(M(L);\mathbb{F}_2)$ represented by $\{a,p,x\}$. Then u^2, v^2 and z^2 are all nonzero, with the sole relation $u^2+v^2+z^2 = 0$ (since $u+v+z$ lifts to an epimorphism onto $\mathbb{Z}/4\mathbb{Z}$). It is then a straightforward exercise using nonsingularity and the 3-fold symmetry to show that $u^2v = u^2z = v^2z = uvz \neq 0$, and so $H^*(M(L);\mathbb{F}_2) \cong \mathbb{F}_2[u,v,z]/\mathcal{I}$, where

$$\mathcal{I} = (u^2+v^2+z^2, u^2v+uv^2, u^2z+uz^2, v^2z+vz^2, u^2v+uvz, v^2z+uvz).$$

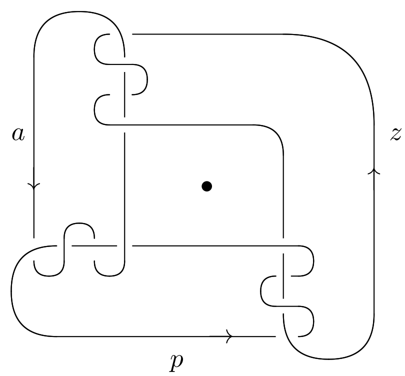

Figure 5. All linking numbers 2.

7. LINKS IN $S^2\tilde{\times}S^1$

Every closed nonorientable 3-manifold may be obtained by surgery on a framed link in (either of) $S^2\tilde{\times}S^1$ or $\mathbb{RP}^2\times S^1$ [7]. We shall construct links in $S^2\tilde{\times}S^1$ from tangles in $D^2\times[0,1]$ with endpoints on the discs $D^2\times\{0\}$ and $D^2\times\{1\}$ by identifying these discs via reflection across a diameter of D^2 to get a link in $Z = D^2\tilde{\times}S^1$, and then attaching another copy of Z to get the double $DZ = Z\cup_{Kb} Z = S^2\tilde{\times}S^1$. We shall assume that the endpoints of the tangle are paired under the reflection, and lie along the diameter fixed by the reflection. (It is not hard to see that every link in $S^2\tilde{\times}S^1$ arises in this way, but we shall not need to prove this.)

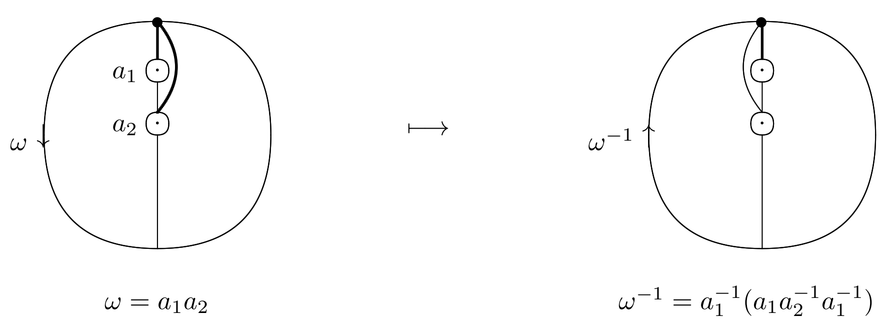

Figure 6. Reflection across the Y-axis

Let L be a link in Z with orientation preserving components. Then each component has an even number of endpoints at each end of $D^2\times[0,1]$, and so bounds a surface in Z which is a union of a spanning surface for a knot or link in $D^2\times[0,1]$

with a number of twisted ribbons. Moreover, we may assume that the twisted ribbons for the various components are all disjoint. Although there is no canonical choice of longitude, we may choose "longitudes" which are $\mathbb{F}_2$-homologically trivial in Z, and we shall let $Y(L)$ be the result of surgery on L in Z to kill these longitudes. Let $\Xi(L) = Y(L) \cup Z$ be the closed 3-manifold obtained by surgery on L in DZ. We shall take the core of the second copy of Z as the standard orientation-reversing loop in $\Xi(L)$.

We may write down a presentation for the fundamental group of the link exterior in terms of meridians and "Wirtinger" relations in the usual way, but there is a slight complication due to the reflection. We assume that the fixed diameter of D^2 is the intersection with the Y-axis, and take the top of this diameter in $D^2 \times \{0\}$ as the basepoint. The meridian for each arc α of the tangle is represented by a loop from the basepoint which passes "in front of" all intermediate arcs, around α, and back over the other arcs to the basepoint. The effect of the reflection is evident from Figure 6, in which the loops a_1 and a_2 are based via the heavy curves, and a_1, a_2 and ω go anticlockwise around the adjacent circles. (Similarly, if $a_1, \dots, a_n$ are generators corresponding to n punctures down the diameter then $\omega = a_1 \dots a_n$ and $\omega^{-1} = a_1^{-1}(a_1 a_2^{-1} a_1^{-1}) \dots (\omega a_n{}^{-1} \omega^{-1})$.)

We shall illustrate this by giving constructions of $S^1 \times \mathbb{RP}^2$ and $S^1 \times Kb$. The fundamental group of the complement of the tangle in Figure 7 has a presentation $\langle a, b, c \mid aba^{-1} = c \rangle$. Identifying the ends gives a knot in Z whose exterior has fundamental group with the presentation

$$\langle a, b, c, t \mid aba^{-1} = c,\ tat^{-1} = c^{-1},\ tbt^{-1} = ca^{-1}c^{-1} \rangle.$$

Adding another copy of Z gives a knot in $S^2 \tilde{\times} S^1$ whose group has presentation

$$\langle a, b, c, t \mid aba^{-1} = c,\ tat^{-1} = c^{-1},\ tbt^{-1} = ca^{-1}c^{-1},\ ab = 1 \rangle,$$

since $\omega = ab$ bounds a fibre of the second copy of $D^2 \tilde{\times} S^1$. If we now perform surgery on the knot to kill $ta^{-1}ta$ the presentation reduces to $\langle a, t \mid ta = at,\ t^2 = 1 \rangle$, and so the fundamental group is $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$.

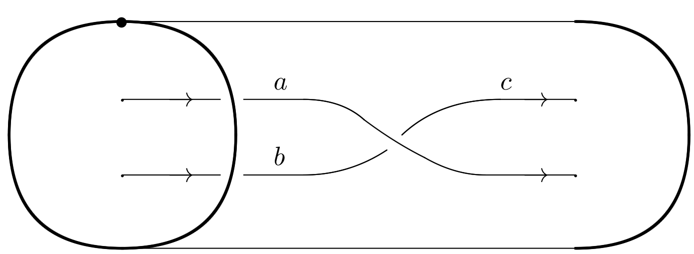

Figure 7. A link for $S^1 \times \mathbb{RP}^2$

From another point of view, we may see directly that surgery on $U_o = \{*\} \times S^1 \subset S^1 \times \mathbb{RP}^2$ gives $S^2 \tilde{\times} S^1$, as follows. Let θ be the reflection of S^2 across an equator. Then $S^2 \tilde{\times} S^1 \cong MT(\theta)$, the mapping torus of θ. Since θ swaps the "polar zones" of S^2, the mapping torus of the restriction of θ to the union of the polar zones is a solid torus $D^2 \times S^1$ (representing $2\times$ a generator of $H_1(S^2 \tilde{\times} S^1; \mathbb{Z})$). The complement of the polar zones is an annulus $\mathbb{A} \cong S^1 \times [-1, 1]$, and θ acts on this product via $\theta(s, x) = (s, -x)$, for all $s \in S^1$ and $-1 \leqslant x \leqslant 1$. Hence $MT(\theta|_{\mathbb{A}}) \cong S^1 \times Mb$.

Since $S^1 \times \mathbb{RP}^2 = S^1 \times Mb \cup S^1 \times D^2$ and $S^2 \tilde{\times} S^1 = MT(\theta|_{\mathbb{A}}) \cup D^2 \times S^1$, the claim follows. Note also that $H_1(S^2 \tilde{\times} S^1; \mathbb{Z})$ is generated by a meridian for U_o.

The fundamental group of the complement of the tangle in Figure 7 has a presentation $\langle a, c, q \mid aqa^{-1} = cqc^{-1} \rangle$. Identifying the ends gives a link in Z whose exterior has fundamental group with the presentation

$$\langle a, b, c, d, q, t \mid aqa^{-1} = cqc^{-1},\ qaq^{-1} = b,\ qcq^{-1} = d,$$
$$tat^{-1} = c^{-1},\ tbt^{-1} = cd^{-1}c^{-1} \rangle.$$

In this case we may add another copy of Z so that $\omega = ab^{-1}$ bounds a disc. Adding the relation $ab^{-1} = 1$ and then killing the longitudes $\ell_q = a^{-1}c$ and $\ell_a = q^{-1}t^{-1}qt$ gives the presentation

$$\langle a, q, t \mid tat^{-1} = a^{-1},\ aq = qa,\ qt = tq \rangle.$$

Hence the resulting 3-manifold is $S^1 \times Kb$.

If instead we give each of the link components a nonzero even framing (so that we kill $q^{-1}t^{-1}qta^{2k}$ and $a^{-1}cq^{2m}$) then the resulting group is a semidirect product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where $A = \left[\begin{smallmatrix} -1 & -2k \\ 2m & 4km+1 \end{smallmatrix}\right]$. This is the group of a nonorientable $\mathbb{S}ol^3$-manifold $MT(A)$ if $km \neq 0$. Taking $k = m = 1$ gives an example with $\rho = 3$, $w^2 = 0$ and $u^3 = v^3 = wuv \neq 0$. (See [4, page 198]. In this reference ρ is the orientation character, called w here.)

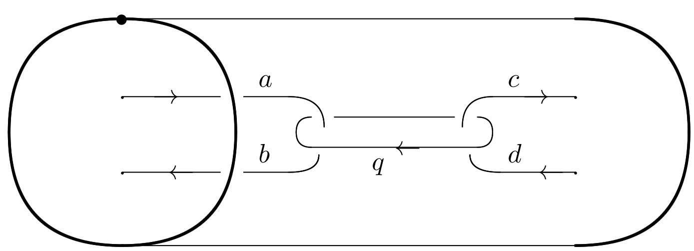

Figure 8. A link for $S^1 \times Kb$ and other $MT(A)$s

8. The nonorientable case

Let $\mathcal{A}^*$ be an MS-algebra with distinguished element w and rank $\rho = \dim \mathcal{A}^1$. The Postnikov-Wu identity is now $wxy = x^2y + xy^2$ for all $x, y \in \mathcal{A}^1$. Hence $wx^2 = 0$ for all x of degree 1. In particular, $w^3 = 0$.

We shall show that $\mathcal{A}^*$ is the $\mathbb{F}_2$-cohomology ring of a closed 3-manifold with orientation character corresponding to w. In our constructions we shall reserve w for the orientation character. In examples with ρ small Poincaré duality considerations may imply that some products x^2 or xy are 0. This is not a hindrance. However we should consider the possible values of cubes x^3.

If $wx = 0$ for all $x \in \mathcal{A}^1$ then $w^2 = 0$, and $\mathcal{A}^*$ may be realized by the connected sum of $S^2 \tilde{\times} S^1$ with an orientable closed 3-manifold. (More generally, if $x \in \mathcal{A}^1$ is nonzero and $xy = 0$ for all $y \in \mathcal{A}^1$ then it suffices to consider a subalgebra of rank $\rho - 1$.) Thus we may assume that $w\bullet : \mathcal{A}^1 \to \mathcal{A}^2$ is nontrivial.

We begin by identifying the smallest MS-algebras of interest, which have $\rho = 2$ or 3 and $w\bullet \neq 0$. Since $w \neq 0$ there is a basis $\{w = x_1, x_2, \ldots, x_\rho\}$ for $\mathcal{A}^1$. We shall modify the basis elements x_i with $i > 1$ so as to simplify the multiplication scheme. Let σ be the rank of $w\bullet : \mathcal{A}^1 \to \mathcal{A}^2$. Then $\sigma > 0$, since $w\bullet \neq 0$. If $w^2 \neq 0$ we may choose the basis so that $\{w^2, wx_2, \dots, wx_\sigma\}$ are linearly independent and $wx_i = 0$ for all $i > \sigma$. If $w^2 = 0$ we assume that $\{wx_2, \dots, wx_{\sigma+1}\}$ are linearly independent and $wx_i = 0$ for all $i > \sigma + 1$.

If $wu \neq 0$ then there is a v such that $wuv \neq 0$, by nonsingularity. The elements wu and wv must be linearly independent, since $wu^2 = 0$. If $w^2 \neq 0$ we may assume that $w^2x_2 \neq 0$. After replacing x_i by $x_i + w$ or $x_i + w + x_2$, if necessary, we may assume that $w^2x_i = wx_2x_i = 0$ for all $i > 2$. If $\sigma > 2$ we may assume that $wx_3x_4 \neq 0$. Further modifications to the basis elements x_i with $i > 4$ then ensure that $wx_3x_i = wx_4x_i = 0$ for all $i > 4$. Iterating this process, we see that σ must be even, and the partial basis $\{w, x_2, \dots, x_\sigma\}$ is partitioned into consecutive pairs whose triple products with w are nonzero, and are the only such products wx_ix_j. A similar argument applies if $w^2 = 0$, and σ is again even. In particular, if $w\bullet \neq 0$ then $\rho > 1$. We may also choose the basis elements x_i with $i > \sigma$ so that $x_i^2x_j \neq 0$ for at most one $j > i$, as in the orientable case, but it is not clear that we can do this in general.

We shall realize these by closed 3-manifolds obtained by surgery on framed links $L \subset S^2\tilde{\times}S^1$. We may assume that such links are disjoint from a standard orientation reversing loop in $S^2\tilde{\times}S^1$. Let $B(L)$ be the complement of a small open tubular neighbourhood of this loop in the closed 3-manifold resulting from surgery on L. Then $\partial B(L) \cong Kb$. Let $N = \overline{S^2 \setminus (\cup_{i=1}^k D_i)}$, where the D_i are disjoint small discs in S^2 which are centred on the equator and invariant under θ. Then $Y = MT(\theta|_N)$ has boundary $\partial Y \cong \sqcup^k Kb$, and we may attach k such 3-manifolds $B(L(i))$ to Y, along the various boundary components. If M is the resulting 3-manifold and we specify an orientation-reversing curve γ_w then the generators of $H_1(M;\mathbb{F}_2)$ apart from the image of γ_w may be chosen to be Kronecker dual to the given basis for the MS-algebra. These arise from loops which lift to the double cover $S^2 \times S^1$ and have product neighbourhoods. We may then meld in products involving the x_i with $wx_i = 0$ by adding copies of Bo and $L_{2,4}$, as appropriate, to the existing link in $S^2\tilde{\times}S^1$.

There are four cases with $1 \leqslant \rho \leqslant 3$ that we need consider. If $\rho = 1$ then $w^2 = 0$, by nonsingularity. This case is realized by $S^2\tilde{\times}S^1$.

If $w^2 \neq 0$ then after replacing $u = x_2$ by $u + w$, if necessary, we may assume that $u^3 = 0$. This case is realized by $S^1 \times \mathbb{RP}^2$.

If $w^2 = 0$ and $\rho = 3$ then we may assume that $wuv \neq 0$, for some $u, v \in \mathcal{A}^1$. If $u^3 = 0$ or $v^3 = 0$ then after replacing v or u by $u + v$, if necessary, we may assume that $u^3 = v^3 = 0$, and that $u^2v \neq 0$. This case is realized by $S^1 \times Kb$.

The other basic case has $w^2 = 0$, $\rho = 3$ and $wuv \neq 0$, and $u^3 = v^3 = (u+v)^3 \neq 0$. This case is realized by the $\mathbb{S}ol^3$-manifold $MT(\left[\begin{smallmatrix} -1 & -2 \\ 2 & 5 \end{smallmatrix}\right])$.

These 3-manifolds shall be our basic building blocks. If Ξ is one of these then in each case $H_1(\Xi;\mathbb{F}_2)$ has a preferred basis consisting of a "standard" orientation-reversing loop and the images of meridians for the components of L, and there is a well-defined "Kronecker dual" basis for $H^1(\Xi;\mathbb{F}_2)$.

9. CONSTRUCTION – SPLICING

In this section we shall show that every (nonorientable) MS-algebra is realized by some (nonorientable) closed 3-manifold. We begin by "splicing" the examples of §8 to realize MS-algebras for which $w\bullet$ is an isomorphism or has kernel $\langle w\rangle$. To handle the general case we shall rely on the following simple observation. Let K be an orientable knot in a 3-manifold M, and let $N_{\mathcal{F}}$ be the result of surgery on K in M with framing $\mathcal{F}$. Let F_1, F_2 and F_3 be mutually transverse closed surfaces in the knot exterior $X = M \setminus K$, and let $x_i \in H^1(N_{\mathcal{F}};\mathbb{F}_2)$ be the Poincaré dual of the class of F_i, for $i \leqslant 3$. Then whether $x_1x_2x_3$ is 0 or not does not depend on the framing $\mathcal{F}$ determining the surgery, since any intersection of submanifolds in the exterior $X = M \setminus K$ is unchanged by any (Dehn) surgery on K.

We shall splice links in $S^2\tilde{\times}S^1$ together as follows. Let D and D' be two small disjoint discs in the interior of D^2 which are invariant under the reflection across the Y-axis, and let $W = \overline{D^2 \setminus D \cup D'}$. Then $Z = W\tilde{\times}S^1 \cup D\tilde{\times}S^1 \cup D'\tilde{\times}S^1$. Let L and L' be links with orientation-preserving components in $D\tilde{\times}S^1$ and $D'\tilde{\times}S^1$, respectively. Then L, L' and $L\sqcup L'$ are links in Z, and there are degree-1 collapses from $\Xi(L\sqcup L')$ onto each of $\Xi(L)$ and $\Xi(L')$. Hence $H^*(\Xi(L);\mathbb{F}_2)$ and $H^*(\Xi(L);\mathbb{F}_2)$ map injectively to $H^*(\Xi(L\sqcup L');\mathbb{F}_2)$. Since $H_1(\Xi(L);\mathbb{F}_2)$ is generated by the images of t and the meridians for L it follows that $H^*(\Xi(L \sqcup L');\mathbb{F}_2)$ is generated by the union of the images of $H^*(\Xi(L);\mathbb{F}_2)$ and $H^*(\Xi(L);\mathbb{F}_2)$. Each component of L has image 0 in $H_1(D\tilde{\times}S^1;\mathbb{F}_2)$, and so bounds a surface in $D\tilde{\times}S^1$. It follows that the duals of a meridian from L and one from L' may be assumed disjoint, and so cup products of the corresponding basis elements for H^1 are 0.

If $w\bullet$ is an isomorphism then $w^2 \neq 0$ and we may partition $\{x_1 = w, \dots, x_\sigma\}$ into consecutive pairs $\{x_{2i-1}, x_{2i}\}$ with $wx_{2i-1}x_{2i} \neq 0$ for all $i \leqslant \frac{1}{2}\sigma$ and $wx_jx_k = 0$ if $k > j+1$ or j is even and $k > j$. (In the latter case we have $x_j^2x_k = x_jx_k^2$, by the Postnikov-Wu identity.) The pair $\{w, x_2\}$ is realized by $S^1 \times \mathbb{RP}^2$, while the other pairs are realized by $S^1 \times Kb$ (if $x_{2i-1}^3 = x_{2i}^3 = 0$) or $MT(\left[\begin{smallmatrix} -1 & -2 \\ 2 & 5 \end{smallmatrix}\right])$ (if $x_{2i-1}^3 = x_{2i}^3 \neq 0$). Thus we may realize $\mathcal{A}^*$ in this case by assembling copies of the links of Figures 7 and 8 (with appropriate framings).

A similar argument applies if $\mathrm{Ker}(w\bullet) = \langle w\rangle$. In this case we need just copies of the link of Figure 8 (with appropriate framings).

In general, let $\tau = \rho - \sigma$ if $w^2 \neq 0$ and $\tau = \rho - \sigma - 1$ if $w^2 = 0$, and adjoin a trivial τ-component link in a ball in $S^2 \times (0,1)$ which is disjoint from the other components. We must now consider the possibility that $x_ix_jx_k \neq 0$, where $1 < i \leqslant j \leqslant k$. We may assume that $k > i$, since the value of x_i^3 is determined by the framing for L_i, and shall write $x = x_i$, $y = x_j$ and $z = x_k$ for simplicity of notation.

If $wxy = wxz = wyz = 0$ then $x^2y = xy^2$, $x^2z = xz^2$ and $y^2z = yz^2$, by the Postnikov-Wu identity, and one of the constructions for the orientable case applies. However, having chosen the basis so as to normalize the nonzero products wuv, we may not be able to reduce the number of possibilities for other triple products. Thus we shall use tangles based on Figures 4 and 5 as well as those of Figures 1 and 3.

If $x^2y = x^2z = y^2z = 0$ then we use move (b) of Figure 1.

If $x^2z = y^2z = 0$ but $x^2y \neq 0$ then we use a tangle based on the link of Figure 3, with z corresponding to the component which is unlinked from each of the other components.

If $x^2y = 0$ but $x^2z = y^2z \neq 0$ then we use a tangle based on the link of Figure 4, with z corresponding to the component which links each of the other components.

If all three of these products are nonzero then we use moves of type (a) as in the link of Figure 5.

Thus we may assume that $wxy \neq 0$. In particular, $x \neq y$. We may also assume that $z \neq x$ or y, and so $wxz = wyz = 0$, by our choice of basis. The Poincaré dual of w in $S^2\tilde{\times}S^1$ is represented by a fibre S^2. It is easily seen that this remains true after the tangle modifications used below, and so these do not disrupt the values of wxy, x^2y or xy^2.

If $x^2z = y^2z = 0$ then we use a move (b).

If $x^2z \neq 0$ and $y^2z = 0$ then we use a tangle based on the link of Figure 3.

If $x^2z = y^2z \neq 0$ then we use move (a) twice.

Finally, we choose the framings of the components lying entirely in $S^2 \times (0,1)$ in accordance with the desired values of the x_i^3s.

10. THE KERNEL OF CUP PRODUCT

If $R = \mathbb{Z}$ or is a field of characteristic $\neq 2$ then cup product induces homomorphisms $c_G^R : \wedge_2 H^1(G;R) \to H^2(G;R)$. We shall write c_G and c_G^p when $R = \mathbb{Z}$ or $\mathbb{F}_p$, respectively. If G is finitely generated then

$$\mathrm{Ker}(c_G) \cong Hom(I(G)/[G,I(G)],\mathbb{Z}).$$

Similarly, if p is an odd prime then

$$\mathrm{Ker}(c_G^p) \cong Hom(G'X^p(G)/[G,G']X^p(G),\mathbb{F}_p).$$

If $p = 2$ then cup product induces $c_G^2 : \odot_2 H^1(G;\mathbb{F}_2) \to H^2(G;\mathbb{F}_2)$ with

$$\mathrm{Ker}(c_G^2) \cong Hom(X^2(G)/[G,X^2(G)]X^4(G),\mathbb{F}_2).$$

In all cases the kernel is determined by $G/\gamma_3G = G/[G,G']$. See [2, 3, 8] for proofs of the above assertions. (In [13] $\mathrm{Ker}(c_G)$ is said to be isomorphic to $Hom(G'/[G,G'],\mathbb{Z})$ "mod torsion". In fact $I(G)/[G,I(G)]$ and $G'/[G,G']$ are commensurable.)

Lemma 3. *Let M be an orientable 3-manifold, let $\pi = \pi_1(M)$ and let F be a field of characteristic $\neq 2$. Then*

(1) if $\beta_1(\pi;F) < 3$ then $c_\pi^F = 0$;
(2) if $\beta_1(\pi;F) = 3$ then c_π^F is either 0 or is an isomorphism.
(3) if $\beta_1(\pi;F) > 3$ then $\mathrm{Ker}(c_\pi^F) \neq 0$.

Proof. If $\alpha \smile \xi \neq 0$ for some $\alpha,\xi \in H^1(M;F)$ then $\alpha \smile \xi \smile \omega \neq 0$, for some $\omega \in H^1(M;F)$, by the nonsingularity of Poincaré duality. Since α,ξ and ω must then be linearly independent, $\beta_1(M;F) \geqslant 3$.

Suppose that $\beta_1(\pi;F) = 3$. Then every element of $\wedge_2 H^1(M;F)$ is a product $v \wedge w$, for if $a \neq 0$ then $ax \wedge y + bx \wedge z + cy \wedge z = a^{-1}(ax - cz) \wedge (ay + bz)$. Hence if c_π^F is not an isomorphism then we may assume that $\alpha \smile \xi = 0$, where $\{\alpha,\xi,\omega\}$ is a basis for $H^1(M;F)$. But then $\{\alpha\wedge\xi, \alpha\wedge\omega, \xi\wedge\omega\}$ is a basis for $\wedge_2 H^1(M;F)$, and $\alpha \smile \xi \smile \omega = 0$. It follows easily from the nonsingularity of Poincaré duality that $c_\pi^F = 0$.

Let $\beta = \beta_1(M;F)$. Then $\beta_2(M;F) = \beta$ also, by Poincaré duality. Hence $\dim_F \mathrm{Ker}(c_\pi^F) \geqslant \binom{\beta}{2} - \beta$, and so $\mathrm{Ker}(c_\pi^F) \neq 0$ if $\beta > 3$. □

If $\pi \cong \mathbb{Z}^3$ then $\beta_1(M;\mathbb{Q}) = 3$ and $c_\pi^{\mathbb{Q}}$ is an isomorphism, while if $\pi \cong F(3)$ then $\beta_1(M;\mathbb{Q}) = 3$ and $c_\pi^{\mathbb{Q}} = 0$.

The case $p = 2$ is different. If $\pi \cong \mathbb{Z}/2\mathbb{Z}$ then $\beta_1(M;\mathbb{F}_2) = 1$ and c_π^2 is an isomorphism. On the other hand, if $\beta = \beta_1(M;\mathbb{F}_2) > 1$ then $\dim_F \mathrm{Ker}(c_\pi^2) \geqslant \binom{\beta+1}{2} - \beta > 0$, and so $\mathrm{Ker}(c_\pi^F) \neq 0$.

Lemma 1 also has implications for the integral case. If $\beta \leqslant 2$ then c must have image in tH, while if $\beta = 3$ then either c has image in tH or it maps $\wedge_1 H^*$ onto H/tH.

11. CUP PRODUCT AND UNIVERSAL COEFFICIENTS

Let p be a prime. The image of $H^1(G;\mathbb{Z})$ in $H^1(G;\mathbb{F}_p)$ is canonical, and the restriction of c_G^p to this image is the *mod-p* reduction of c_G. However if G^{ab} has p-torsion this does not fully determine c_G^p.

We may construct 3-manifold examples illustrating this as follows. Let $M = M(Bo;p)$ and let $\pi = \pi_1(M)$. Then $\pi^{ab} \cong (\mathbb{Z}/p\mathbb{Z})^3$ and $X^p(\pi) = \pi'$. Hence c_π^p is injective, by the criterion of [2]. On the other hand, if $N = \#^3 L(p, q_i)$ (for some q_i such that $(q_i, p) = 1$) and $G = \pi_1(N) \cong *^3(\mathbb{Z}/p\mathbb{Z})$ then $c_G^p = 0$. Clearly $H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z})$ as rings. Moreover, we may choose the parameters q_i so that $\ell_M \cong \ell_N$. Thus H, γ, D_2 and ℓ do not determine the mod-p cohomology ring.

The group Γ_q with presentation

$$\langle x, y, z \mid [x, y] = z^q,\ zx = xz,\ zy = yz\rangle.$$

is the fundamental group of a $\mathbb{N}il^3$-manifold, and $\beta_1(\Gamma_q) = 2$, so $c_{\Gamma_q}^{\mathbb{Q}} = 0$, by Lemma 1. On the other hand, $I(\Gamma_q) = \zeta\Gamma_q \cong \mathbb{Z}$ is generated by the image of z. Hence $[\Gamma_q, I(\Gamma_q)] = 1$, but $\Gamma_q' = qI(\Gamma_q)$, as it is generated by the image of z^q. Thus if p is an odd prime which divides q then $\Gamma_q' < X^p(\Gamma_q)$ and $c_{\Gamma_q}^p$ is injective.

More explicitly, let f and g be the homomorphisms from Γ_q to $\mathbb{Z}$ defined by $f(x) = 1$, $f(y) = f(z) = 0$ and $g(y) = 1$, $g(x) = g(z) = 0$, and let $\overline{f}_p, \overline{g}_p : \Gamma_q \to \mathbb{F}_p$ be their mod-(p) reductions. Then the image of $f \cup g$ in $H^2(\Gamma_q;\mathbb{Q})$ is 0, but $\overline{f}_p \cup \overline{g}_p \neq 0$ [8, Theorem 1], for each p dividing q. Thus $f \cup g$ generates the torsion subgroup of $H^2(\Gamma_q;\mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/q\mathbb{Z}$. In this case $c_{\Gamma_q}^{\mathbb{Q}} = 0$, so $\mu = 0$, but $c_{\Gamma_q} \neq 0$, $c_{\Gamma_q}^p \neq 0$ and $\nu_p \neq 0$.

Let $h : \Gamma_q \to \mathbb{F}_p$ be defined by $h(x) = h(y) = 0$ and $h(z) = 1$. Reduction mod-(p) maps $H^2(\Gamma_q;\mathbb{Z})$ onto $H^2(\Gamma_q;\mathbb{F}_p)$, and the latter group has basis

$$\{\overline{f}_p \cup h, \overline{g}_p \cup h, \overline{f}_p \cup \overline{g}_p\}.$$

The first two elements are reductions of cohomology classes of infinite order. However, h does not lift to a homomorphism to $\mathbb{Z}$, and these classes are not cup products of elements of $H^1(\Gamma_q;\mathbb{Z})$. Thus we may have $\mathrm{Im}(c) \leqslant tH$, and so cannot treat the torsion and torsion-free parts separately.

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